STUDENT NUMBER: ________________

STUDENT NAME: __________________

THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2008

MATHEMATICS

EXTENSION 1

Teacher Responsible: Mrs P Singh

General Instructions:

- Reading time – 5 minutes
- Working time – 2 hours
- This paper contains 7 questions.
- ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- **Start each question in a new booklet.**
- A table of standard integrals is supplied at the back of this paper.
- Marks may be deducted for careless, untidy or badly arranged work.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

Trial Extension 1 – 2008

Question 1 (12 marks) **Marks**

a) The polynomial $P(x) = x^3 + ax^2 + 2x - 4$ has a remainder of -7 when divided by $x + 2$. Find the value of a. **1**

b) Differentiate $e^{2x} \sin x$ **2**

c) Find the acute angle, to the nearest degree, between the lines

$$2x + y = 4 \text{ and } x - y = 2$$ **2**

d) Evaluate $\displaystyle\int_0^2 \frac{dx}{4+x^2}$ **2**

e) Using the substitution $u = 2x + 1$ or otherwise, find $\displaystyle\int_0^1 \frac{4x}{2x+1}\,dx$. **3**

f) Find the co-ordinates of the point P which divides the line joining $A(-3, 4)$ and $B(2, -8)$ externally in the ratio 2:5. **2**

Question 2 (12 marks)

a) Prove the identity: $\dfrac{\cos x - \cos 2x}{\sin 2x + \sin x} = \operatorname{cosec} x - \cot x$ **3**

b) Find all angles for θ, where $0 \le \theta \le 2\pi$ for which $\sqrt{3}\cos\theta - \sin\theta = 1$. **4**

c) The function $h(x)$ is given by $h(x) = \sin^{-1} x + \cos^{-1}(x)$ for $-1 \le x \le 1$.

i) show that $h'(x) = 0$ **1**

ii) sketch the graph of $y = h(x)$ **2**

d) Find $\dfrac{dy}{dx}$ if $y = \tan^{-1}(\sin x)$ **2**

Question 3 (12 marks) **Marks**

a) A cup of hot cappuccino at temperature T^0 Celsius loses heat when placed in a cooler environment. It cools according to the law $\frac{dT}{dt} = -k\left(T - T_0\right)$ where time, t is the time elapsed in minutes and T_0 is the temperature of the environment in degrees Celsius.

i) Show that $T = T_0 + Ce^{-kt}$ **1**

ii) A cup of cappuccino at 100^0C is placed in an environment at -20^0C for 3 minutes and then cools to 70^0C. Find *k*, in exact form. **2**

iii) The same cup of cappuccino at 70^0C is then placed in an environment at 20^0C, assuming *k* stays the same, find the temperature, to the nearest degree, of the cappuccino after a further 15 minutes. **4**

b) i) Show that $x = 2$ is a zero of $x^3 - 4x^2 + 8$ **1**

ii) Hence find all the real zeros of $x^3 - 4x^2 + 8$, leaving your answers in exact form. **2**

iii) Hence solve the inequality: $\frac{4}{x-2} \le x$ **2**

Question 4 (12 marks) **Marks**

a) Solve $2^{2x+1} - 5(2^x) + 2 = 0$ **3**

b) Find the coefficient of x^3 in $\left(3x^2 + \frac{1}{x}\right)^9$ **3**

c) A function is defined as $f(x) = 1 + e^{2x}$.

i) Write down the range of $f(x)$. **1**

ii) Given $f^{-1}(x)$ is the inverse function for $f(x)$, show that

$$f^{-1}(x) = \frac{1}{2}\ln(x-1)$$ **2**

iii) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing all key features. **3**

Question 5 (12 marks) **Marks**

a) A particle moves in a straight line with Simple Harmonic Motion. At time t seconds, its displacement x metres from a fixed point O, is given by:

$$x = 5\sin\frac{\pi}{2}\left(t+\frac{1}{3}\right)$$

i) Show that $\ddot{x} = \frac{-\pi^2}{4}x$ **2**

ii) State the period and the amplitude of the motion. **2**

b) The acceleration of a particle moving in a straight line is given by:

$$\frac{d^2x}{dt^2} = \frac{-72}{x^2},$$

where x metres is the displacement from the origin after t seconds. Initially the particle is 9 metres to the right of the origin with a velocity of 4m per second.

i) Show that the velocity v of the particle in terms of x is $v = \frac{12}{\sqrt{x}}$.

Explain why v is always positive for the given initial conditions. **5**

ii) Find an expression for t in terms of x. **3**

Question 6 (12 marks) **Marks**

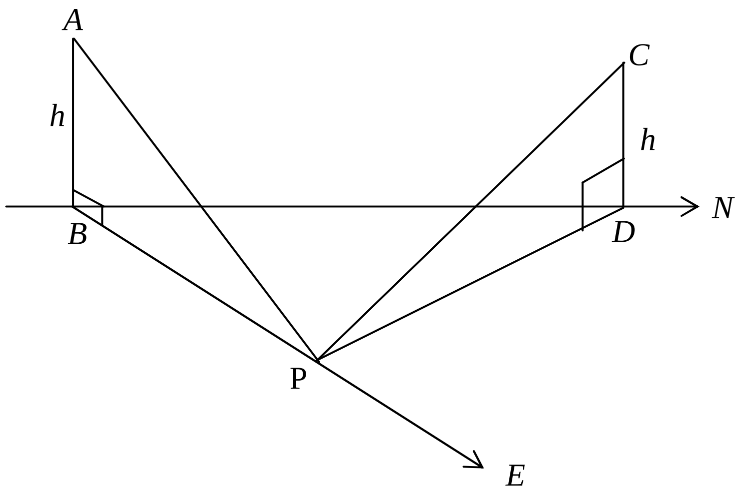
a) i) Show that the equation of the tangent to the parabola $x^2 = 16y$ at any point $P\left(8t,\ 4t^2\right)$ on it is $y = tx - 4t^2$. **2**

ii) Show that the equation of the line *r* through the focus *S* of the parabola which is perpendicular to the focal chord through *P* is

$$\left(t^2 - 1\right)y + 2tx = 4\left(t^2 - 1\right)$$ **2**

iii) Prove that the locus of the point of intersection of the line *r* and the tangent at *P* is a horizontal line. **3**

b) *AB* and *CD* are two towers of equal height (*h*). *CD* is due north of *AB*. From a point *P* on the same horizontal plane as the feet *B* and *D* of the towers, and bearing due east of the tower *AB*, the angles of elevation of *A* and *C*, the tops of the towers, are 47° and 31° respectively. If the distance between the towers is 88m, find the height of the towers to the nearest metre. **5**



Question 7 (12 marks) **Marks**

a) Prove by mathematical induction that
$1.2^0 + 2.2^1 + 3.2^2 + ... + n.2^{n-1} = 1 + (n-1)2^n$ for $n \geq 1$. **4**

b) During a soccer tournament, Juan is standing 25m away from the goal line. He kicks a soccer ball off the ground at an angle of 30° to the horizontal with an initial velocity of V m per sec. The ball hits the top bar which is 2.4 m directly above the goal line. Neglecting air resistance and assuming that acceleration due to gravity is 10 m/s^2, find:

i) The horizontal and vertical components of displacement of the ball in terms of the initial velocity V. **4**

ii) The Cartesian equation of the motion for the path of the ball. **1**

iii) The initial velocity of the ball, correct to 1 decimal place. **3**

END OF EXAMINATION

THGS Soln Trial 2008 Ext 1

Q1 (12 marks)

(a) $-8 + 4a - 4 - 4 = -7$

$4a = 9$

$\therefore a = 9/4$ (1)

(b) $y = e^{2x} \sin x$ (u, v)

$y' = e^{2x} \cos x + \sin x \; 2e^{2x}$

$= e^{2x}(\cos x + 2\sin x)$ (2)

(c) $y = -2x + 4 \quad \therefore m_1 = -2$

$y = x - 2 \quad m_2 = 1$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = |3|$

$\therefore \theta = 72^\circ$ (2)

(d) $\int_0^2 \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(0)$

$= \frac{1}{2} \times \frac{\pi}{4} - \frac{1}{2} \times 0$

$= \frac{\pi}{8}$ (2)

(e) $\int_0^1 \frac{4x}{2x+1}$

$u = 2x + 1 \implies 2x = u - 1$

$du = 2dx$

when $x = 1$, $u = 3$

$x = 0$, $u = 1$

$= \int_1^3 \frac{u-1}{u}\,du$

$= \int_1^3 1 - \frac{1}{u}\,du$

$= [u - \log_e u]_1^3$

$= (3 - \log_e 3) - (1 - \log_e 1)$

$= 2 - \log_e 3$ (3)

(f) $P\left(\frac{-2\times 2 + 5\times -3}{-2+5}, \frac{-2\times -8 + 5\times 4}{-2+5}\right)$

$= \left(-\frac{19}{3}, \frac{36}{3}\right)$

$= \left(-6\frac{1}{3}, 12\right)$ (2)

Q2. 12 marks

(a) LHS $= \frac{\cos x - \cos 2x}{\sin 2x + \sin x} = \frac{\cos x + 2\cos^2 x + 1}{2\sin x\cos x + \sin x}$

$= \frac{(1 + 2\cos x)(1 - \cos x)}{\sin x(2\cos x + 1)}$

$= \frac{1 - \cos x}{\sin x}$

$= \frac{1}{\sin x} - \frac{\cos x}{\sin x}$

$= \operatorname{cosec} x - \cot x$

QED (3)

(b) $\sqrt{3}\cos\theta - \sin\theta = r\cos(\theta + \alpha)$

$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

$r\cos\alpha = \sqrt{3} \qquad r\sin\alpha = 1$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}} \qquad r^2 = (\sqrt{3})^2 + 1^2$

$\therefore \alpha = \pi/6 \qquad r = 2$

$\therefore 2\cos\left(\theta + \frac{\pi}{6}\right) = 1$

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3} - \frac{\pi}{6}, \; \frac{5\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{6}$ or $\frac{3\pi}{2}$ (4)

(c) (i) $h'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$. as reqd. (1)

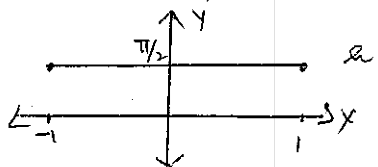
(ii) $\therefore$ function $h(x)$ has a gradient of zero

$\therefore$ function is a straight line of the form $y = a$ where a is the constant.

when $x = 0$, $h(x) = \frac{\pi}{2}$

$\therefore h(x) = \frac{\pi}{2}$ for $-1 \le x \le 1$

(2)



$h(x) = \sin^{-1}(x) + \cos^{-1}(x)$

(d) Let $u = \sin x$

$\frac{du}{dx} = \cos x$.

$\therefore y = \tan^{-1} u$

$y' = \frac{1}{1+u^2}$

$\therefore \frac{dy}{dx} = \cos x \times \frac{1}{1+\sin^2 x}$

$= \frac{\cos x}{1+\sin^2 x}$ (2)

Soln Trial 2008 Ext 1.

Question 3 (12 marks)

3(a) (i) $\frac{dT}{dt} = -kCe^{-kt}$ but $\frac{dT}{dt} = -k(T - T_0)$

$\therefore -k(T - T_0) = -kCe^{-kt}$ (1)

$\therefore T = T_0 + Ce^{-kt}$

(ii) At $t = 3$, $T = 70$ $T_0 = -20$

$70 = -20 + 120e^{3k}$ at $t = 0$, $T = 100$

$e^{3k} = \frac{9}{12}$ $\therefore 100 = -20 + A$

$\therefore k = \frac{1}{3}\ln\frac{3}{4}$ $\therefore A = 120$ (2)

(iii) let $t = 0$, when cup is placed in enviro at 20°C.

$T = 20 + Be^{kt}$

at $t = 0$, $T = 70°$

$70 = 20 + Be^{0}$

$\therefore B = 50$

$\therefore T = 20 + 50e^{kt}$

$\therefore T = 20 + 50e^{15(\frac{1}{3}\ln\frac{3}{4})}$

$\doteq 31.86°$

$\simeq 32°C$ (4)

(b) (i) $P(+2) = 8 - 16 + 8 = 0$ $\therefore$ a zero. (1)

(ii)
$$
\begin{array}{r|l}
 & x^2 - 2x - 4 \\
x-2 & x^3 - 4x^2 + 8 \\
 & -\ x^3 - 2x^2 \\
 & -2x^2 \\
 & (-)\ -2x^2 + 4x \\
 & -4x + 8 \\
 & (-)\ -4x + 8 \\
\end{array}
$$

$\therefore P(x) = (x-2)(x^2 - 2x - 4)$

$x = \frac{2 \pm \sqrt{4 + 16}}{2}$

$= \frac{2 \pm 2\sqrt{5}}{2}$ (2)

$\therefore x = 2,\ 1 \pm \sqrt{5}$

Q3

b. (iii) $\frac{4}{x-2} \leq x$

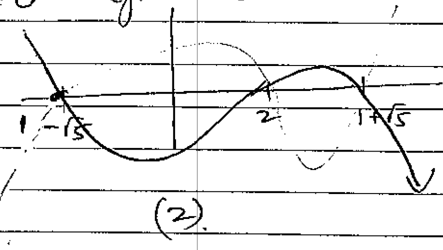
$4(x-2) \leq x(x-2)^2$

$4(x-2) - x(x-2)^2 \leq 0$

$(x-2)(4 - x^2 + 2x) \leq 0$

$(x-2)(-x^2 + 2x + 4) \leq 0$ (from (ii))

$x = 2$ $x = 1 \pm \sqrt{5}$



$SS = \{x : x \geq 1 + \sqrt{5},\ 1 - \sqrt{5} \leq x < 2\}$ (2)

Q4 (12 marks)

(b) $\left(3x^2 + \frac{1}{x}\right)^9$

$T_{k+1} = {}^9C_k\,(3x^2)^{9-k}\left(\frac{1}{x}\right)^k$

$= {}^9C_k \,.\, 3^{9-k}\, x^{18-2k} \,.\, x^{-k}$

$= {}^9C_k \,.\, 3^{9-k} \,.\, x^{18-3k}$ (3)

$\therefore$ coeff of x^3

$\Rightarrow 18 - 3k = 3$

$3k = 15$

$k = 5$

$\therefore$ coeff $= {}^9C_5 \times 3^{9-5}$

$= 126 \times 3^4$

$= 10206$

(c) $f(x) = 1 + e^{2x}$

i) $y > 1$ (1)

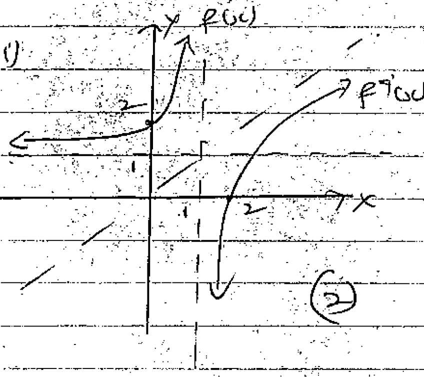
ii) $x = 1 + e^{2y}$

$x - 1 = e^{2y}$ (3)

$\ln(x-1) = 2y$

$\therefore y = \frac{1}{2}\ln(x-1)$

(iii)



(2)

(d) $2^{2x+1} - 5(2^x) + 2 = 0$

$2(2^{2x}) - 5(2^x) + 2 = 0$

let $u = 2^x$

$\therefore 2u^2 - 5u + 2 = 0$

$(2u-1)(u-2) = 0$

$u = \frac{1}{2},\ u = 2$

$\therefore x = -1,\ 1$ (3)

Q5

(a) (i) $x = 5\sin\frac{\pi}{2}\left(t + \frac{1}{3}\right)$

$\dot{x} = \frac{5\pi}{2}\cos\frac{\pi}{2}\left(t+\frac{1}{3}\right)$

$\ddot{x} = \frac{-5\pi^2}{4}\sin\frac{\pi}{2}\left(t+\frac{1}{3}\right)$

$\therefore \ddot{x} = -\frac{\pi^2}{4}x$ (2)

(ii) Ampl $= 5$, period $= \frac{2\pi}{\pi/2} = 4$. (2)

(b) (i) $\frac{d^2x}{dt^2} = \frac{-72}{x^2}$ $t = 0, x = 9, v = 4$

$\frac{d\,\frac{1}{2}v^2}{dx} = \frac{-72}{x^2}$

$\frac{1}{2}v^2 = \frac{-72x^{-1}}{-1} + c_1$

$\frac{1}{2}v^2 = \frac{72}{x} + c_1$

$\left.\begin{array}{l} x = 9 \\ v = 4 \end{array}\right\}$ $\frac{1}{2}(16) = \frac{72}{9} + c_1$

$\therefore c_1 = 0$

$\therefore \frac{1}{2}v^2 = 72/x$

$v^2 = 144/x$

$v = \pm\frac{12}{\sqrt{x}}$

But particle starts at 9 + is travelling to the right at 4m/s $\therefore$ velocity is +ve as $v = \frac{12}{\sqrt{x}}$.

v is never 0 $\Rightarrow \frac{12}{\sqrt{x}} \neq 0$ as $x \neq 0$ (5)

(ii) $\frac{dx}{dt} = \frac{12}{\sqrt{x}}$ at $t = 0$, $x = 9$

$\frac{dt}{dx} = \sqrt{x}/12$ $\therefore 0 = \frac{9^{3/2}}{18} + C_2$

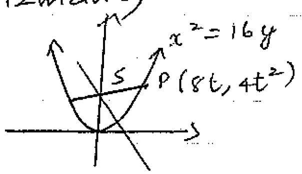
$t = \frac{x^{3/2}}{18} + C_2$ $\therefore C_2 = -3/2$

$\therefore t = \frac{x^{3/2}}{18} - 3/2$ (3)

$t = \frac{\sqrt{x^3}}{18} - 3/2$

Q6 (12 marks)

(a) (i)



i) $x^2 = 16y$

$2x = 16\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x}{8}$

$= \frac{8t}{8}$

$= t$.

Eqn of tan: $y - 4t^2 = t(x - 8t)$

$y = tx - 8t^2 + 4t^2$ ✓✓ (2)

$y = tx - 4t^2$ - - ①

sub. for tx

ii) $S(0,4)$ $P(8t, 4t^2)$

grad $SP = \frac{4t^2-4}{8t}$

$= \frac{t^2-1}{2t}$

$\therefore$ grad ⊥ $= \frac{-2t}{t^2-1}$

eqn: $y - 4 = \frac{-2tx}{t^2-1}$

$t^2y + 4 - 4t^2 - y = -2tx$ (2)

$(t^2-1)y + 2tx = 4t^2 - 4$

R: $(t^2-1)y = 2tx = 4(t^2-1)$ - - ②.

iii) sub ① in ②

$(t^2-1)(tx - 4t^2) + 2tx = 4t^2 - 4$ (1 sub)

[can eliminate x + sub get to $y = 4$]

$t^3x - 4t^4 - tx + 4t^2 + 2tx = 4t^2 - 4$

$t^3x + tx = 4t^4 - 4$

$x(t^3 + t) = 4t^4 - 4$ ①

$x = \frac{4(t^4-1)}{t(t^2+1)} = \frac{4(t^2-1)(t^2+1)}{t(t^2+1)} = \frac{4(t^2-1)}{t}$ ③ (3)

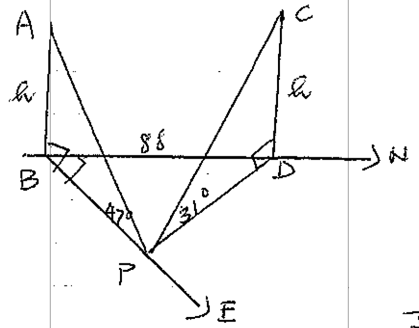
sub in ① y: $t \cdot \frac{4(t^2-1)}{t} - 4t^2$

$= 4t^2 - 4 - 4t^2$

$= -4$ ①

$\therefore$ locus is line $y = 4$ ①

(b)



Let height of towers be h.

$BP = h\cot 47°$ ✓

$PD = h\cot 31°$ ✓

In ΔBDP,

$h^2\cot^2 31° = 88^2 + h^2\cot^2 47°$

$h^2(\cot^2 31° - \cot^2 47°) = 88^2$

$h^2 = \frac{88^2}{\cot^2 31° - \cot^2 47°}$ ✓

$= \frac{88^2}{\tan^2 59° - \tan^2 43°}$ ✓

$\doteqdot 4075.2708$

$\therefore h = 63.837$. ✓ (5)

$\therefore$ height of towers is 64 m (to accuracy

Q 7. (12 marks)

(a) Prove $\sum_{r=1}^{n} r.2^{r-1} = 1 + (n-1)2^n$

ie prove $1.2^0 + 2.2^1 + 3.2^2 + \cdots + n\,2^{n-1} = 1 + (n-1)2^n$

Step 1 Prove true for n = 1

LHS $= 1.2^0 = 1$

RHS $1 + 0.2^1 = 1$

∴ true for n = 1

(4)

Step 2. Assume true for n = k

ie $1.2^0 + 2.2^1 + \cdots k\,2^{k-1} = 1 + (k-1)2^k$

Step 3 Prove true for n = k+1

ie $1.2^0 + \cdots + k.2^{k-1} + (k+1)2^k$

$= 1 + k.2^{k+1}$

LHS $1 + (k-1)2^k + (k+1)2^k$

$= 1 + k.2^k - 2^k + k2^k + 2^k$

$= 1 + 2k.2^k$

$= 1 + k.2^1\,2^k$

$= 1 + k.2^{k+1}$

= RHS.

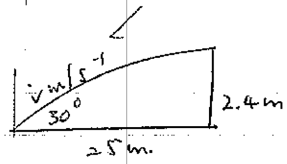
∴ true for n = k+1 if true for n = k.

(4)

Step 4 Concl.

We have proved stmt true for n = k+1 if true for n = k.
But stmt is true for n = 1 ∴ True for n = 2, 3, 4, n ∈ J.

7(b)



(i) Horizontal:

$\ddot{x} = 0$

$\dot{x} = c$

$t = 0, \dot{x} = \frac{\sqrt{3}v}{2}$

$\therefore \dot{x} = \frac{\sqrt{3}v}{2}$

$x = \frac{\sqrt{3}}{2}vt + c$

when $t = 0$, $x = 0$

$x = \frac{\sqrt{3}}{2}vt$

Vertical

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t = 0$, $\dot{y} = \frac{v}{2}$

$\dot{y} = -10t + \frac{v}{2}$

$y = -5t^2 + \frac{vt}{2} + c$

when $t = 0$, $y = 0$

$\therefore y = -5t^2 + \frac{vt}{2}$ (4)

(ii) $t = \frac{2x}{\sqrt{3}v}$

$\therefore y = -5\left(\frac{2x}{\sqrt{3}v}\right)^2 + \frac{v}{2}\left(\frac{2x}{\sqrt{3}v}\right)$

$y = \frac{-20x^2}{3v^2} + \frac{x\sqrt{3}}{3}$ (1)

(iii) when $x = 25$, $y = 2.4$

$2.4 = -\frac{20 \times 25^2}{3v^2} + \frac{25\sqrt{3}}{3}$

$7.2v^2 = -12500 + 25\sqrt{3}v^2$

$v^2 = \frac{12500}{25\sqrt{3} - 7.2}$

$= 346.248$

$\therefore v = 18.6$ m/s.

(3)